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EMBEDDING WEAKLY COMPACT SETS INTO HILBERT SPACE[†]

BY

Y. BENYAMINI[††] AND T. STARBIRD[†††]

ABSTRACT

We give an example of a weakly compact set in a Banach space, which does not embed topologically as a weakly compact subset of Hilbert space. We also show that a weakly compact set embeds in a super-reflexive space iff it embeds in Hilbert space.

It was proved in [2] that every weakly compact subset of a Banach space is (affinely) homeomorphic to a weakly compact subset of a reflexive Banach space. In this paper we study the existence of homeomorphic embedding into a super-reflexive space. We show that the existence of such an embedding is equivalent to the existence of an embedding into a Hilbert space. The main part of the paper is an example of a weakly compact set which does not embed in Hilbert space (in its w topology).

If Γ is a set, $c_0(\Gamma)$ will denote the Banach space of all functions f on Γ such that for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma : |f(\gamma)| > \varepsilon\}$ is finite. The norm in $c_0(\Gamma)$ is the sup norm. By $l_2(\Gamma)$ we shall denote the Hilbert space of all square summable elements in $c_0(\Gamma)$ with the usual norm. The weak topology on a weakly compact subset of $c_0(\Gamma)$ is exactly the topology of point-wise convergence.

We shall also consider weakly compact subsets of $c_0(\Gamma)$ that will consist only of characteristic functions of finite sets. In this case we shall identify a set with its characteristic function, and consider the sets as being elements in $c_0(\Gamma)$.

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A compact Hausdorff space will be called an Eberlein compact if it is homeomorphic to a weakly compact subset of a Banach space. The main structure theorem on Eberlein compacts is due to Amir and Lindenstrauss [1]:

Every Eberlein compact is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ *for some* Γ.

The cardinality of a set A will be denoted by $\# A$.

An Eberlein compact K will be called uniform if there is an embedding of K into $c_0(\Gamma)$ and a function $N(\varepsilon)$ such that for all k in K and for all $\varepsilon > 0$, $\# \{\gamma : |k(\gamma)| > \varepsilon\} < N(\varepsilon)$.

The density character of the Banach space X (i.e. the minimal cardinality of a dense subset in X) will be denoted by $\dim(X)$.

THEOREM. *The following conditions on a compact Hausdorff space* K *are equiva ent*:

(1) K *embeds as a* w*-compact subset in a Hilbert space.*

(2) K *is a uniform Eberlein compact.*

(3) K *embeds as a* w*-compact subset in a super-reflexive space.*

PROOF. Clearly (1) implies (2) and (3).

(2) $\Rightarrow$ (1): Assume the function $N(\varepsilon)$ and the embedding of K into $c_0(\Gamma)$ are given. Since K is weakly compact it is bounded in $c_0(\Gamma)$ and we can assume that $\|k\| \leqq 1$ for all k in K. Let $f:[-1,1] \to [-1,1]$ be a continuous, strictly monotone and odd function such that for all $n, f(n^{-1}) \leqq (2^n N(1/(n+1))^{-1/2}$ and define a mapping $\phi: K \to l_2(\Gamma)$ by $\phi(k)(\gamma) = f(k(\gamma))$. Fix any k in K and let $A_n = \{\gamma : (n+1)^{-1} < |k(\gamma)| \leqq n^{-1}\}$. Then

$$\|\phi(k)\|^2_{l_2(\Gamma)} = \sum_\gamma |f(k(\gamma))|^2 = \sum_n \sum_{\gamma \in A_n} |f(k(\gamma))|^2 \leqq \sum_n \# A_n \cdot |f(n^{-1})|^2$$
$$\leqq \sum 2^{-n} = 1 .$$

Thus ϕ maps K into $l_2(\Gamma)$ and it is clearly 1-1 and ω-continuous.

(3) $\Rightarrow$ (2): Let X be a super-reflexive space such that $\dim(X) = \mu$, and let K be a weakly compact subset of X.

We can assume that K is contained in the unit ball of X. We shall prove the theorem by transfinite induction on μ.

Let $\{P_\alpha\}_{\alpha<\mu}$ be a "long sequence of projections" in X as in [1], and let $X_\alpha = (P_{\alpha+1} - P_\alpha)X$, and $K_\alpha = (P_{\alpha+1} - P_\alpha)(K)$. Since $\dim(X_\alpha) < \mu$ for all α we can find, by the induction hypothesis, a homeomorphism ϕ_α of K_α into the unit ball of some Hilbert space H_α with orthonormal basis $\{e_i\}_{i \in \Gamma_\alpha}$.

It follows from a theorem of James [3] that there is a $p < \infty$, such that for every y in X, $(\Sigma \| (P_{\alpha+1} - P_\alpha) y \|^p)^{1/p} \leqq 2 \| y \|$. Define now a mapping $\phi : K \to (\Sigma \oplus H_\alpha)_{l_p}$ by $[\phi(k)](\alpha) = \phi_\alpha((P_{\alpha+1} - P_\alpha) k)$. It is clear that ϕ is 1-1 and ω-continuous and thus a homeomorphism.

Fix now any k in K and $\varepsilon > 0$. Then $\# \{\alpha : \| (P_{\alpha+1} - P_\alpha) k \| > \varepsilon\} < 2\varepsilon^{-p}$ and for each such α, $\# \{i \in \Gamma_\alpha : \phi(k) > \varepsilon\} < \varepsilon^{-2}$. And thus $\phi(K)$ is a uniform Eberlein compact with function $N(\varepsilon) = 2\varepsilon^{-(p+2)}$.

Before presenting the example of an Eberlein compact which does not embed in a Hilbert space, we shall need some simple lemmas.

LEMMA 1. *Let $\{A_\alpha\}_{\alpha \in \mathfrak{U}}$ be an uncountable collection of finite subsets of a set Γ. Then there exists an uncountable subset $\mathfrak{U}_1$ of $\mathfrak{U}$ and a finite subset A of Γ, such that for each $\alpha \in \mathfrak{U}_1$, $A_\alpha = A \cup B_\alpha$ and the B_α are pairwise disjoint.*

For a proof see [4], p. 87.

LEMMA 2. *Let Γ be a set and $K = \{A\}$ a collection of finite subsets of Γ such that*

(1) *If $A \in K$ and $B \subset A$ then $B \in K$.*

(2) *There is no infinite increasing chain in K.*

Then K is a weakly compact set in $c_0(\Gamma)$ (when we identify sets in K with their characteristic functions).

PROOF. Let S be a limit point of K and let F be a finite subset of S. Since the topology is given by pointwise convergence, there exists an A in K such that $F \subset A$, and by (1) we have that $F \in K$. By (2) this means that S is finite, and thus also $S \in K$.

LEMMA 3. *Let K be a weakly compact subset of Hilbert space, and let $D \subset K$ be a discrete set with a unique limit point k. Then for every $d \in D$ there exists a relatively open subset V_d of K such that :*

(1) *For all $d \in D$, d is in V_d.*

(2) *There is a countable partition $D = \cup D_n$ such that if $d_1, \cdots, d_{n+1}$ are distinct elements in D_n, then $\bigcap_{j=1}^{n+1} V_{d_j} = \emptyset$.*

PROOF. We can assume without loss of generality that K is the unit ball of the Hilbert space and that k is the origin. We shall denote by (x, y) the inner product in the space.

Fix $d_0 \in D$. Since the origin is the only weak limit point of D, there are at most countably many $d \in D$ such that $(d, d_0) \neq 0$. Thus, if we define an equivalence relation on D by $d \sim e$ if there are $d_1 = d, d_2, \cdots, d_m = e$ in D such that $(d_j, d_{j+1}) \neq 0$, the equivalence classes are at most countable. Clearly elements in different equivalence classes are orthogonal. We can thus decompose $D = \cup E_m$ where the elements in each E_m are mutually orthogonal. By decomposing each E_m we can arrive at a partition $D = \cup D_n$ where the elements in each D_n are mutually orthogonal and such that if d is in D_n, $\| d \|^4 > n^{-1}$. We now define for $d \in D_n$, $V_d = \{k \in K : (k, d)^2 > n^{-1}\}$. Now if $d_1, \cdots, d_{n+1} \in D_n$ are distinct and $k \in \bigcap_{j=1}^{n+1} V_{d_j}$ we would have $1 \geqq \| k \|^2 \geqq \Sigma_1^{n+1} (k, d_j)^2 \geqq (n+1)/n > 1$, a contradiction.

THE EXAMPLE. Let $\Gamma = [0,1] \times \Pi_{n=2}^{\infty}\{1, \cdots, n\}$. We write an element in Γ as $(r, n_2, n_3, \cdots)$ where $r \in [0,1]$ and $1 \leqq n_j \leqq j$. We denote by P_m the projection defined by $P_m(r, n_2, n_3, \cdots) = (n_2, \cdots, n_m)$. For each $m \geqq 1$ we define now a set K_m of subsets of Γ of cardinality m: K_1 is the set of all singletons in Γ. For each sequence $n_2, \cdots, n_{m-1}$, we define $K_{n_2, \cdots, n_{m-1}} = \{\{d_1, \cdots, d_m\} \subset \Gamma : P_m(d_j) = (n_2, \cdots, n_{m-1}, j)\}$ and we define $K_m = \cup K_{n_2, \cdots, n_{m-1}}$.

We define now K as follows: $K = \{A \subset \Gamma : A \subset B$ for some B in $\cup K_m\}$.

We shall show that K is a weakly compact set in $c_0(\Gamma)$ which does not embed in Hilbert space.

To show that K is weakly compact we shall use Lemma 2. The first condition is satisfied by the definition of K. Since sets in K_m have cardinality exactly m, the second condition will follow once we show that if $A_1 \in K_{m_1}, A_2 \in K_{m_2}$ and $m_1 \neq m_2$, then $\#(A_1 \cap A_2) \leqq 1$. But assume $m_1 > m_2$; then all points in A_1 have the same m_2th coordinate, while different points in A_2 have different m_2th coordinates.

We now consider $D = K_1 \subset K$. It is easy to check that D is a discrete set in the weak topology with a unique limit point—the empty set. Let $\{V_d\}$ be a collection of open sets in K with $d \in V_d$ for each $d \in D$. We shall show that if $D_n \subset D$ satisfy the condition that for each distinct $d_1, \cdots, d_{n+1} \in D_n$, $\bigcap_{j=1}^{n+1} V_{d_j} = \emptyset$, then $\cup D_n \neq D$. By Lemma 3 this will imply that K does not embed in Hilbert space.

We can assume without loss of generality that each V_d is a basic open set, i.e. there is a finite subset A_d of Γ such that $A \in V_d$ iff $A \in K$ and there exists a set $B \in V_d$ with $A \cap A_d = B \cap A_d$. We can also assume that $d \in A_d$ for each d.

PROPOSITION. *Fix $m \geqq 2$ and $n_2, \cdots, n_{m-1}$. Then at least one of the sets $D_{m-1} \cap P_m^{-1}(n_2, \cdots, n_{m-1}, j), 1 \leqq j \leqq m$, is at most countable.*

Before we prove the proposition, let us show how to finish the proof. We define inductively a sequence $1 \leqq n_j \leqq j$ as follows: We let n_2 be such that

$D_1 \cap P_2^{-1}(n_2)$ is at most countable. Having already defined $n_2, \cdots, n_{m-1}$, we let n_m be such that $D_{m-1} \cap P_m^{-1}(n_2, \cdots, n_m)$ is at most countable. But then

$$\left(\bigcup_{m=1}^{\infty} D_m \right) \cap \left(\bigcap_{m=2}^{\infty} P_m^{-1}(n_2, \cdots, n_m) \right) \subset \bigcup_{m=2}^{\infty} (D_{m-1} \cap P_m^{-1}(n_2, \cdots, n_m))$$

is at most countable. Since $\bigcap_{m=2}^{\infty} P_m^{-1}(n_2, \cdots, n_m)$ is uncountable (there is no restriction on r), $\cup D_m \neq D$.

Proof of the Proposition. Assume that for each $1 \leqq j \leqq m$ there is an uncountable subset Γ_j of $D_{m-1} \cap P_m^{-1}(n_2, \cdots, n_{m-1}, j)$. By Lemma 1 we can assume that there are finite sets $A_1, \cdots, A_m$ such that if $d \in \Gamma_j$ then $A_d = A_j \cup B_d$ and the sets $\{B_d\}_{d \in \Gamma_j}$ are pairwise disjoint. Since the set $A_1 \cup \cdots \cup A_m$ is finite we can also assume that $d \notin A_1 \cup \cdots \cup A_m$ for each $d \in \Gamma_1 \cup \cdots \cup \Gamma_m$. We now choose $d_1, \cdots, d_m$, with $d_j \in \Gamma_j$ such that $d_j \notin A_{d_i}$ if $i \neq j$. We pick any $d_1 \in \Gamma_1$. Since B_{d_1} is finite and the sets $\{B_d\}_{d \in \Gamma_2}$ are pairwise disjoint we can find $d_2 \in \Gamma_2$ such that $d_1 \notin B_{d_2}$ and $d_2 \notin B_{d_1}$. Continuing inductively we can thus find $d_j \in \Gamma_j$ such that if $i \neq j$, $d_i \notin B_{d_j}$. Since we also have that $d_i \notin A_1 \cup \cdots \cup A_m$ we really have that $d_i \notin A_{d_j}$ if $i \neq j$.

We claim that $\{d_1, \ , d_m\} \in \bigcap_{j=1}^{m} V_{d_j}$ which contradicts the assumption on D_{m-1}. Indeed, $\{d_1, \cdots, d_m\}$ is really in K (in fact it is in $K_{n_2, \cdots, n_{m-1}}$) and since $d_j \in V_{d_j}$ and V_{d_j} depends only on A_{d_j}, the equality $\{d_1, \cdots, d_m\} \cap A_{d_j} = \{d_j\} \cap A_{d_j}$ implies that also $\{d_1, \cdots, d_m\} \in V_{d_j}$.

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Yale University
New Haven, Conn., U.S.A.

and

University of California
Berkeley, Cal., U.S.A.